

Some characterizations for Markov processes as mixed renewal processes

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July 14, 2014

Abstract

For a mixed renewal process (MRP for short) with mixing parameter a random vector we prove under mild assumptions that it has the multinomial property if and only if it is a Markov process or equivalently that it is a mixed Poisson process with mixing parameter a random variable. We provide a simple example showing that our assumptions are essential for these equivalences to hold true. As a consequence the invariance of the Markov property under a certain change of measures follows. As an application we single out some concrete examples of non-trivial probability spaces, admitting MRPs and allowing us to check whether a given MRP has the Markov property or not.

MSC 2010: Primary 60G55 ; secondary 60K05, 60J27, 28A50, 60A10, 91B30.

Key Words: *mixed renewal process, Markov property, mixed Poisson process, disintegration.*

Introduction

Mixed renewal processes are widely used, especially in Risk Theory for modelling the occurrence of rare events (cf. e.g. [12]).

To the best of our knowledge, a first definition of MRPs with mixing distribution traces back to Huang's paper [5], Definition 3. There he proved that under mild assumptions a MRP with mixing distribution has the Markov property if and only if it is a mixed Poisson process (MPP for short) with mixing distribution (see [5], Theorem 3).

An alternative way to model MRPs within a class of counting processes is to assume the existence of a random vector on the same probability space such that under conditioning on this random vector, the counting process behaves like an ordinary renewal process (see [8], Definition 3.2). A MRP according to Huang's definition is always a MRP according to Definition 3.2 of [8] (see [8], Theorem 4.9).

In Section 2 we improve Huang's result by showing that under mild assumptions a MRP with mixing parameter a random vector is a Markov process if and only if it is

a MPP with mixing parameter a random variable if and only if it has the multinomial property (see Proposition 2.7). In Theorem 2.11, our main result, the above Proposition is generalized for the wider class of extended MRPs (see Definition 2.3, **(b)**), being more proper than the class of Definition 2.3, **(a)** for the applications (see Examples 3.2).

A basic tool for the proof of Theorem 2.11 is the existence of an appropriate disintegration $\{P_\theta\}_{\theta \in \mathbb{R}^d}$ of a given probability measure P , and the reduction of a MRP under P to a renewal process under the disintegrating measures P_θ . Note that the existence of such a disintegration is guaranteed for a wide class of probability spaces used in applied Probability Theory (see the Remark following Definition 2.1).

As a consequence the invariance of the Markov property, as well as that of the multinomial property, under the change of the measure P into P_θ for almost all $\theta \in \mathbb{R}^d$ is obtained.

In Section 3, a method for the construction of non-trivial probability spaces admitting extended MRPs is given, providing concrete examples of probability spaces and (extended) MRPs satisfying the assumptions of the main result and allowing us to check whether a (extended) MRP has the Markov property or not.

Further applications of our main result, concerning the equivalence of the existing definitions of MPPs, will be given in our forthcoming paper [9].

1 Preliminaries

By \mathbb{N} is denoted the set of all natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The symbol \mathbb{R} stands for the set of all real numbers, while $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$. If $d \in \mathbb{N}$, then \mathbb{R}^d denotes the Euclidean space of dimension d .

Given a probability space (Ω, Σ, P) , a set $N \in \Sigma$ with $P(N) = 0$ is called a **P -null set** (or a null set for simplicity). The family of all P -null sets is denoted by Σ_0 . Given a measurable space (\mathcal{Y}, T) , for any two Σ - T -measurable maps $X, Y : \Omega \longrightarrow \mathcal{Y}$ we write $X = Y$ P -a.s. if $\{X \neq Y\} \in \Sigma_0$.

If $A \subseteq \Omega$, then $A^c := \Omega \setminus A$, while χ_A denotes the indicator function of the set A . The σ -algebra generated by a family \mathcal{G} of subsets of Ω is denoted by $\sigma(\mathcal{G})$. A σ -algebra \mathcal{A} is **countably generated** if there exists a countable family \mathcal{G} of subsets of Ω such that $\mathcal{A} = \sigma(\mathcal{G})$.

For any Hausdorff topology \mathfrak{T} on Ω by $\mathfrak{B}(\Omega)$ is denoted the **Borel σ -algebra** on Ω , i.e. the σ -algebra generated by \mathfrak{T} . By $\mathfrak{B} := \mathfrak{B}(\mathbb{R})$, $\overline{\mathfrak{B}} := \mathfrak{B}(\overline{\mathbb{R}})$, $\mathfrak{B}_d := \mathfrak{B}(\mathbb{R}^d)$ and $\mathfrak{B}_{\mathbb{N}} := \mathfrak{B}(\mathbb{R}^{\mathbb{N}})$ is denoted the Borel σ -algebra of subsets of \mathbb{R} , $\overline{\mathbb{R}}$, \mathbb{R}^d and $\mathbb{R}^{\mathbb{N}}$, respectively, while $\mathcal{L}^1(P)$ stands for the family of all real-valued P -integrable functions on Ω . Functions that are P -a.s. equal are not identified. A **random variable** on Ω is a Σ -measurable function $X : \Omega \longrightarrow \mathbb{R}$. An **extended random variable** is a Σ - $\overline{\mathfrak{B}}$ -measurable function from Ω into $\overline{\mathbb{R}}$. A **d -dimensional random vector** on Ω is a Σ - \mathfrak{B}_d -measurable function $X : \Omega \longrightarrow \mathbb{R}^d$.

Given two probability spaces (Ω, Σ, P) and (\mathcal{Y}, T, Q) as well as a Σ - T -measurable map $X : \Omega \longrightarrow \mathcal{Y}$ we denote by $\sigma(X) := \{X^{-1}(B) : B \in T\}$ the σ -algebra generated by X .

Setting $T_X := \{B \subseteq \mathcal{Y} : X^{-1}(B) \in \Sigma\}$ for any given Σ - T -measurable map X from Ω into \mathcal{Y} , we clearly get that $T \subseteq T_X$. Denote by $P_X : T_X \longrightarrow \mathbb{R}$ the image measure of P under X . The restriction of P_X to T is denoted again by P_X . For any random vector $X : \Omega \longrightarrow \mathbb{R}^d$ ($d \in \mathbb{N}$) the notation $P_X = \mathbf{K}(\theta)$ will be state that X is distributed according to the law $\mathbf{K}(\theta)$, where $\theta \in \tilde{\Theta}$ and $\tilde{\Theta}$ is a parametric space. In particular, $\mathbf{P}(\theta)$ and $\mathbf{Exp}(\theta)$, where θ is positive parameter, stand for the law of Poisson and exponential distribution, respectively (cf. e.g. [10]).

If $X \in \mathcal{L}^1(P)$ and \mathcal{F} is a σ -subalgebra of Σ , then each function $Y \in \mathcal{L}^1(P \upharpoonright \mathcal{F})$ satisfying for each $A \in \mathcal{F}$ the equality $\int_A X dP = \int_A Y dP$ is said to be a **version of the conditional expectation** of X with respect to (or given) \mathcal{F} and it will be denoted by $\mathbb{E}_P[X \mid \mathcal{F}]$. For $X := \chi_B \in \mathcal{L}^1(P)$ with $B \in \Sigma$ we set $P(B \mid \mathcal{F}) := \mathbb{E}_P[\chi_B \mid \mathcal{F}]$.

By $(\Omega \times \mathcal{Y}, \Sigma \otimes T, P \otimes Q)$ is denoted the product probability space of (Ω, Σ, P) and (\mathcal{Y}, T, Q) , and by π_Ω and $\pi_{\mathcal{Y}}$ the canonical projections from $\Omega \times \mathcal{Y}$ onto Ω and \mathcal{Y} , respectively.

Given two measurable spaces (Ω, Σ) and (\mathcal{Y}, T) , a **T - Σ -Markov kernel** is a function k from $T \times \Omega$ into \mathbb{R} satisfying the following conditions:

- (k1) The set-function $k(\bullet, \omega)$ is a probability measure on T for any fixed $\omega \in \Omega$.
- (k2) The function $\omega \longmapsto k(B, \omega)$ is Σ -measurable for any fixed $B \in T$.

Let be given a Σ - T -measurable map from Ω into \mathcal{Y} and a σ -subalgebra \mathcal{F} of Σ . A **conditional distribution of X over \mathcal{F}** is a T - \mathcal{F} -Markov kernel k satisfying for each $B \in T$ condition

$$k(B, \bullet) = P(X^{-1}(B) \mid \mathcal{F})(\bullet) \quad P \upharpoonright \mathcal{F} - a.s..$$

Such a Markov kernel will be denoted by $P_{X \mid \mathcal{F}}$. In particular, if (Ξ, Z) is a measurable space, Θ is a Σ - Z -measurable map from Ω into Ξ and $\mathcal{F} := \sigma(\Theta)$, then the function $P_{X \mid \Theta} := P_{X \mid \sigma(\Theta)}$ is called a **conditional distribution of X given Θ** . Clearly, for every T - Z -Markov kernel k , the map $K(\Theta)$ from $T \times \Omega$ into \mathbb{R} defined by means of

$$K(\Theta)(B, \omega) := (k(B, \bullet) \circ \Theta)(\omega) \quad \text{for any } B \in T \quad \text{and } \omega \in \Omega$$

is a T - $\sigma(\Theta)$ -Markov kernel. In particular, for $(\mathcal{Y}, T) = (\mathbb{R}^d, \mathfrak{B}_d)$, for $d \in \mathbb{N}$, its associated probability measures $k(\bullet, \theta)$ for $\theta = \Theta(\omega)$ with $\omega \in \Omega$ are distributions on \mathfrak{B} and so we may write $\mathbf{K}(\theta)(\bullet)$ instead of $k(\bullet, \theta)$. Consequently, in this case $K(\Theta)$ will be denoted by $\mathbf{K}(\Theta)$. For any σ -subalgebra \mathcal{F} of Σ , we say that two T - \mathcal{F} -Markov kernels k_i , for $i \in \{1, 2\}$, are $P \upharpoonright \mathcal{F}$ -**equivalent** and we write $k_1 = k_2 \ P \upharpoonright \mathcal{F}$ -a.s., if there exists a P -null set $N \in \mathcal{F}$ such that for any $\omega \notin N$ and $B \in T$ the equality $k_1(B, \omega) = k_2(B, \omega)$ holds true.

From now on (Ω, Σ, P) is a probability space, while (\mathcal{Y}, T) and (Ξ, Z) are measurable spaces, all of them arbitrary but fixed.

2 Characterizations via mixed Poisson processes and the multinomial property

A sequence $T := \{T_n\}_{n \in \mathbb{N}_0}$ of random variables on Ω is a **claim arrival process**, if there exists a null set $\Omega_T \in \Sigma_0$ such that for all $\omega \in \Omega \setminus \Omega_T$ we have $T_0(\omega) = 0$ and $T_{n-1}(\omega) < T_n(\omega)$ for all $n \in \mathbb{N}$. The sequence $W := \{W_n\}_{n \in \mathbb{N}}$, given by $W_n := T_n - T_{n-1}$ for each $n \in \mathbb{N}$ outside of a null set $\Omega_W := \Omega_T$, is then called the **claim interarrival process induced by** the claim arrival process T (cf. e.g. [10], Section 1.1, page 6). Conversely, given a claim interarrival process W we define the **induced claim arrival process** T by setting $T_n := \sum_{k=1}^n W_k$ for all $n \in \mathbb{N}_0$ outside of $\Omega_T := \Omega_W$. A family $N := \{N_t\}_{t \in \mathbb{R}_+}$ of extended random variables on Ω is called a **counting** or a **claim number process** if there exists a null set $\Omega_N \in \Sigma_0$ such that the process N restricted on $\Omega \setminus \Omega_N$ takes values in $\mathbb{N}_0 \cup \{\infty\}$, has right-continuous paths, presents jumps of size (at most) one, vanishes at $t = 0$ and increases to infinity. Each claim arrival process T induces a counting process N by means of

$$N_t(\omega) := \sum_{n=1}^{\infty} \chi_{\{T_n \leq t\}}(\omega) \quad \text{for all } t \in \mathbb{R}_+$$

and $\omega \notin \Omega_N := \Omega_T$ (cf. e.g. [10], Theorem 2.1.1). Conversely, if N is a counting process with exceptional null set Ω_N then the sequence T defined by

$$T_n(\omega) := \inf\{t \in \mathbb{R}_+ : N_t(\omega) = n\} \quad \forall n \in \mathbb{N}_0$$

for all $\omega \notin \Omega_T := \Omega_N$ is the **claim arrival process induced by** N .

Recall that a family of random variables $\{X_i\}_{i \in I}$ on (Ω, Σ, P) is **P -conditionally (stochastically) independent** given a σ -algebra $\mathcal{F} \subseteq \Sigma$, if for each $n \in \mathbb{N}$ with $n \geq 2$ we have

$$P\left(\bigcap_{j=1}^n \{X_{i_j} \leq x_{i_j}\} \mid \mathcal{F}\right) = \prod_{k=1}^n P(\{X_{i_k} \leq x_{i_k}\} \mid \mathcal{F}) \quad P \upharpoonright \mathcal{F} - \text{a.s.}$$

whenever i_1, \dots, i_n are distinct members of I and $(x_{i_1}, \dots, x_{i_n}) \in \mathbb{R}^n$.

The family $\{X_i\}_{i \in I}$ is **P -conditionally identically distributed** over \mathcal{F} , if

$$P(F \cap X_i^{-1}(B)) = P(F \cap X_j^{-1}(B))$$

whenever $i, j \in I$, $F \in \mathcal{F}$ and $B \in \mathcal{T}$. For simplicity, we write *P -conditionally i.i.d.* instead of *P -conditionally independent and P -conditionally identically distributed*.

Furthermore, if Θ is a Σ - Z -measurable map from Ω into Ξ , we say that $\{X_i\}_{i \in I}$ is **P -conditionally (stochastically) independent or identically distributed given Θ** , if it is conditionally independent or identically distributed over the σ -algebra $\sigma(\Theta)$.

Throughout what follows, unless it is stated otherwise, we put $(\Xi, Z) = (\mathbb{R}^d, \mathfrak{B}_d)$, for $d \in \mathbb{N}$, and we simply write “conditionally” in the place of “conditionally given Θ ”

whenever conditioning refers to Θ . Moreover, $N := \{N_t\}_{t \in \mathbb{R}_+}$ is a counting process, $T := \{T_n\}_{n \in \mathbb{N}_0}$ is its induced claim arrival process, $W := \{W_n\}_{n \in \mathbb{N}}$ is its induced claim interarrival process and without loss of generality we may and do assume that $\Omega_N = \Omega_T = \Omega_W = \emptyset$, and that N has zero probability of explosion.

Definition 2.1 Let Q be a probability measure on T . A family $\{P_y\}_{y \in \mathcal{Y}}$ of probability measures on Σ is called a **disintegration** of P over Q if

(d1) for each $D \in \Sigma$ the map $y \mapsto P_y(D)$ is T -measurable;

(d2) $\int P_y(D)Q(dy) = P(D)$ for each $D \in \Sigma$.

If $f : \Omega \rightarrow \mathcal{Y}$ is an inverse-measure-preserving function (i.e. $P(f^{-1}(B)) = Q(B)$ for each $B \in T$), a disintegration $\{P_y\}_{y \in \mathcal{Y}}$ of P over Q is called **consistent** with f if, for each $B \in T$, the equality $P_y(f^{-1}(B)) = 1$ holds for Q -almost every $y \in B$.

Remark. If Σ is countably generated and P is perfect (see [2], p. 291, for the definition), then there always exists a disintegration $\{P_y\}_{y \in \mathcal{Y}}$ of P over Q consistent with any inverse-measure-preserving map f from Ω into \mathcal{Y} providing that T is countably generated (see [2], Theorems 6 and 3). So, in most cases appearing in applications (e.g. Polish spaces) disintegrations as above always exist.

A counting process N is said to be a **Poisson process with parameter $\theta > 0$** (or a P -PP(θ) for short) if it has stationary independent increments and $P_{N_t} = \mathbf{P}(\theta t)$ holds for all $t \in (0, \infty)$.

Definition 2.2 (cf. e.g. [10], page 87) A counting process N is a **mixed Poisson process** on (Ω, Σ, P) with parameter a random variable Θ such that $P_\Theta((0, \infty)) = 1$ (or a P -MPP(Θ) for short), if it has P -conditionally stationary independent increments and

$$\forall t \in (0, \infty) \quad P_{N_t|\Theta} = \mathbf{P}(\Theta t) \quad P \upharpoonright \sigma(\Theta) - \text{a.s.}$$

The following two definitions of MRPs being in line with the definition of a mixed Poisson process (MPP for short) with parameter Θ seem to be the natural ones, since among others they involve explicitly the *structural parameter* Θ .

Definitions 2.3 (a) A counting process N is said to be a **mixed renewal process** on (Ω, Σ, P) with parameter an \mathbb{R}^d -valued random vector Θ on Ω , and claim interarrival time conditional distribution $\mathbf{K}(\Theta)$ (or a $(P, \mathbf{K}(\Theta))$ -MRP for short), if the induced claim interarrival process W is P -conditionally independent and

$$\forall n \in \mathbb{N} \quad P_{W_n|\Theta} = \mathbf{K}(\Theta) \quad P \upharpoonright \sigma(\Theta) - \text{a.s.}$$

In particular, if there exists a $\theta_0 > 0$ with $P(\Theta = \theta_0) = 1$, then N is a *P -renewal process* with claim interarrival time distribution $\mathbf{K}(\theta_0)$ (or a $(P, \mathbf{K}(\theta_0))$ -RP for short), and if $(\Xi, Z) = (\mathbb{R}, \mathfrak{B})$, $P_\Theta((0, \infty)) = 1$ and $\mathbf{K}(\Theta) = \mathbf{Exp}(\Theta)$ $P \upharpoonright \sigma(\Theta)$ -a.s. then a $(P, \mathbf{K}(\Theta))$ -MRP becomes a P -MPP(Θ) (see [6], Proposition 4.5).

(b) More generally N is called **an extended MRP** on (Ω, Σ, P) with parameters Θ and h , where Θ is an \mathbb{R}^d -valued random vector Θ on Ω , and h is a \mathbb{R}^k -valued \mathfrak{B}_d - \mathfrak{B}_k -measurable function on \mathbb{R}^d , for $k \in \mathbb{N}$, if the induced claim interarrival process W is P -conditionally independent and

$$\forall n \in \mathbb{N} \quad P_{W_n|\Theta} = \mathbf{K}(h(\Theta)) \quad P \upharpoonright \sigma(\Theta) - \text{a.s.}$$

Huang's definition for MRPs (see [5], Definition3) does not involve a structural parameter Θ .

Before we formulate the basic result of this section we need the next auxiliary lemma.

Lemma 2.4 *Let $\{P_\theta\}_{\theta \in \mathbb{R}^d}$ be a disintegration of P over P_Θ consistent with Θ , let N be a counting process and let $h : \mathbb{R}^d \rightarrow \mathbb{R}^k$ ($k \in \mathbb{N}$) be a \mathfrak{B}_d - \mathfrak{B}_k -measurable function. Assume that there exists a P_Θ -null set $L \in \mathfrak{B}_d$ such that the restriction $h \upharpoonright L^c$ is injective. Put $\tilde{\Theta} := h \circ \Theta$, $g := (h \upharpoonright L^c)^{-1} : h(L^c) \rightarrow L^c$, and $M^c := g^{-1}(L^c)$. For any $\tilde{\theta} \in \mathbb{R}^k$ and $A \in \Sigma$ define*

$$\hat{g}(\tilde{\theta}) := \begin{cases} g(\tilde{\theta}) & \text{if } \tilde{\theta} \notin M \\ 0 \in \mathbb{R}^d & \text{if } \tilde{\theta} \in M \end{cases}$$

and

$$Q_{\tilde{\theta}}(A) := \begin{cases} (P_\bullet(A) \circ g)(\tilde{\theta}) & \text{if } \tilde{\theta} \notin M \\ P(A) & \text{if } \tilde{\theta} \in M \end{cases}.$$

Then the following holds true:

(i) the family $\{Q_{\tilde{\theta}}\}_{\tilde{\theta} \in \mathbb{R}^k}$ is a disintegration of P over $P_{\tilde{\Theta}}$ consistent with $\tilde{\Theta}$;

(ii) for any $n \in \mathbb{N}$ the equivalence

$$\forall \theta \notin L \quad (P_\theta)_{W_n} = \mathbf{K}(h(\theta)) \iff \forall \tilde{\theta} \notin M \quad (Q_{\tilde{\theta}})_{W_n} = \mathbf{K}(\tilde{\theta})$$

is fulfilled;

(iii) for every $\theta \notin L$ the process W is P_θ -independent if and only if for every $\tilde{\theta} \notin M$ it is $Q_{\tilde{\theta}}$ -independent.

Proof. First note that the function g is $\mathfrak{B}(h(L^c))$ - $\mathfrak{B}(L^c)$ -measurable (cf. e.g. [1] Proposition 8.3.5 together with Theorem 8.3.7); hence \hat{g} is \mathfrak{B}_k - \mathfrak{B}_d -measurable. As a consequence, we get that $M \in \mathfrak{B}_k$ and $P_{\tilde{\Theta}}(M) = 0$.

Ad (i): Clearly, $\{Q_{\tilde{\theta}}\}_{\tilde{\theta} \in \mathbb{R}^k}$ is a family of probability measure on Σ satisfying condition (d1). Condition (d2) follows by (d2) for $\{P_\theta\}_{\theta \in \mathbb{R}^d}$.

To show that $\{Q_{\tilde{\theta}}\}_{\tilde{\theta} \in \mathbb{R}^k}$ is consistent with $\tilde{\Theta}$, let $A \in \Sigma$ and $B \in \mathfrak{B}_k$ be arbitrary. Putting $E := h^{-1}(B)$ we get

$$(\hat{g})^{-1}(E) = (B \cap M^c) \cup (\hat{g})^{-1}(E \cap L);$$

hence

$$\begin{aligned}
\int_B Q_{\tilde{\theta}}(A) P_{\tilde{\Theta}}(d\tilde{\theta}) &= \int_{B \cap M^c} (P_{\bullet}(A) \circ g)(\tilde{\theta}) P_{\tilde{\Theta}}(d\tilde{\theta}) + \int_{B \cap M} Q_{\tilde{\theta}}(A) P_{\tilde{\Theta}}(d\tilde{\theta}) \\
&= \int_{g^{-1}(E \cap L^c)} (P_{\bullet}(A) \circ g)(\tilde{\theta}) P_{\tilde{\Theta}}(d\tilde{\theta}) \\
&= \int_{E \cap L^c} P_{\theta}(A) P_{\Theta}(d\theta) = P(A \cap \Theta^{-1}(E \cap L^c)) \\
&= P(A \cap (\tilde{\Theta})^{-1}(B)),
\end{aligned}$$

where the fourth equality follows by the consistency of $\{P_{\theta}\}_{\theta \in \mathbb{R}^d}$ with Θ . This completes the proof of (i).

Ad (ii): Let us fix on arbitrary $n \in \mathbb{N}$ and $A \in \Sigma$ such that $A := W_n^{-1}(B)$ for $B \in \mathfrak{B}((0, \infty))$. Assume that for all $\theta \notin L$ we have $(P_{\theta})_{W_n} = \mathbf{K}(h(\theta))$. Then for any $\tilde{\theta} \notin M$ we get $\theta := g(\tilde{\theta}) \notin L$, implying that

$$\begin{aligned}
(Q_{\tilde{\theta}})_{W_n}(B) &= Q_{\tilde{\theta}}(A) = (P_{\bullet}(A) \circ g)(\tilde{\theta}) = P_{g(\tilde{\theta})}(A) = P_{\theta}(A) \\
&= (P_{\theta})_{W_n}(B) = \mathbf{K}(h(\theta))(B) = \mathbf{K}(\tilde{\theta})(B).
\end{aligned}$$

For the inverse implication, assume that for any $\tilde{\theta} \notin M$ we have $Q_{\tilde{\theta}}(A) = \mathbf{K}(\tilde{\theta})$. Then for any $\theta \notin L$ we get $\tilde{\theta} := g^{-1}(\theta) \notin M$; hence

$$\begin{aligned}
(P_{\theta})_{W_n}(B) &= P_{\theta}(A) = P_{g(\tilde{\theta})}(A) = (P_{\bullet}(A) \circ g)(\tilde{\theta}) = Q_{\tilde{\theta}}(A) \\
&= (Q_{\tilde{\theta}})_{W_n}(B) = \mathbf{K}(\tilde{\theta})(B) = \mathbf{K}(h(\theta))(B).
\end{aligned}$$

Assertion (iii) follows in a similar way. □

We recall the following definitions, needed for the formulation of Remark 2.6 and Proposition 2.7.

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Definitions 2.5 A counting process N is said to have

(a) the **multinomial property**, if the identity

$$P\left(\bigcap_{j=1}^m \{N_{t_j} - N_{t_{j-1}} = \kappa_j\}\right) = \frac{n!}{\prod_{j=1}^m \kappa_j!} \prod_{j=1}^m \left(\frac{t_j - t_{j-1}}{t_m}\right)^{\kappa_j} \cdot P(\{N_t = n\})$$

holds for all $m \in \mathbb{N}$, $t_0, t_1, \dots, t_m \in \mathbb{R}_+$ such that $0 = t_0 < t_1 < \dots < t_m$ and all $\kappa_1, \dots, \kappa_m \in \mathbb{N}_0$ such that $n = \sum_{j=1}^m \kappa_j$ (cf. e.g. [11], page 2);

(b) the **Markov property**, or it is said to be a **Markov process**, if the identity

$$P\left(\{N_{t_{m+1}} = n_{m+1}\} \mid \bigcap_{j=1}^m \{N_{t_j} = n_j\}\right) = P(\{N_{t_{m+1}} = n_{m+1}\} \mid \{N_{t_m} = n_m\})$$

holds for all $m \in \mathbb{N}$, $t_1, \dots, t_m, t_{m+1} \in (0, \infty)$, and $n_1, \dots, n_m, n_{m+1} \in \mathbb{N}_0$ such that $t_1 < \dots < t_m < t_{m+1}$ and $P(\bigcap_{j=1}^m \{N_{t_j} = n_j\}) > 0$ (cf. e.g. [10], page 44)

Remark 2.6 The following result is well known (cf. e.g. [5], Theorem 2) but we write it exactly in the form, that we need. rem1

Let $\theta \in \mathbb{R}^d$ be fixed, and let N be a $(P, \mathbf{K}(\theta))$ -RP. For any $t \in \mathbb{R}_+$ put $F_\theta(t) := P(\{W_n \leq t\})$ for all $n \in \mathbb{N}$. Assume that the function F_θ is continuously differentiable on $(0, \infty)$, $0 < F'_\theta(t) < C$ for each $t > 0$, where C is a positive constant, which may depend of θ , and that $p(\theta) := \lim_{t \rightarrow 0} F'_\theta(t)$ is positive. Then the following assertions are equivalent:

- (i) N has the Markov property;
- (ii) N is a P -PP($p(\theta)$).

It is well known that if N is a P -MPP(Θ) then it satisfies the Markov property (cf. e.g. [10], Theorem 4.2.3). However, the trivial counting process N defined by means of $N_t := [t]$ for every $t \in \mathbb{R}_+$, where by $[t]$ is denoted the integer part of t , is a Markov $(P, \mathbf{K}(\theta_0))$ -RP but not any Poisson process. This raises the question, under which conditions a Markov $(P, \mathbf{K}(\Theta))$ -MRP is a P -MPP(Θ) ? An answer to this question is provided by the following result.

Proposition 2.7 Let $\{P_\theta\}_{\theta \in \mathbb{R}^d}$ a disintegration of P over P_Θ consistent with Θ and let N be a $(P, \mathbf{K}(\Theta))$ -MRP. For any $\theta \in \mathbb{R}^d$ and $t \in \mathbb{R}_+$ put $F_\theta(t) := P_\theta(\{W_n \leq t\})$ for all $n \in \mathbb{N}$ and assume that there exists a P_Θ -null set $L_1 \in \mathfrak{B}_d$ such that for any $\theta \notin L_1$ the following conditions are satisfied prop1

- (*) the function F_θ is continuously differentiable on $(0, \infty)$ and there exists a function $C \in \mathcal{L}^1(P_\Theta)$ with $0 < F'_\theta(t) < C(\theta)$ for each $t > 0$, and
- (**) the real valued function p on L_1^c defined by means of $p(\theta) := \lim_{t \rightarrow 0} F'_\theta(t)$ is positive and injective.

Then the following assertions are equivalent:

- (i) N has the P -multinomial property;
- (ii) N has the P -Markov property;
- (iii) there exists a random variable $\check{\Theta}$ on Ω such that N is a P -MPP($\check{\Theta}$).

Proof. The implication $(i) \implies (ii)$ follows by [11], Theorem 4.2, while the implication $(iii) \implies (i)$ is well known for any random variable on Ω with values in $(0, \infty)$ in the place of $\check{\Theta}$ (cf. e.g. [10], Lemma 4.2.2). Note also that the implication $(i) \implies (ii)$ holds true without the assumptions (*) and (**).

Ad $(ii) \implies (iii)$: Assume that N has the Markov property.

(a) There exists a P_Θ -null set $L_2 \in \mathfrak{B}_d$ such that N is a $(P_\theta, \mathbf{K}(\theta))$ -RP for all $\theta \notin L_2$. a1

In fact, since N is a $(P, \mathbf{K}(\Theta))$ -MRP we may apply [8], Proposition 3.8, to obtain (a).

(b) For every $t > 0$ and $n \in \mathbb{N}_0$ condition $P(\{N_t = n\}) > 0$ holds true.

In fact, first notice that by applying [6], Lemma 3.5, for any $n \in \mathbb{N}_0$ and any $t > 0$ we obtain

$$P(\{N_t = n\}) = \mathbb{E}_{P_\Theta} [P_\bullet(\{N_t = n\})].$$

Therefore, it is sufficient to show that for every $t > 0$ and $n \in \mathbb{N}_0$ condition $P_\theta(\{N_t = n\}) > 0$ holds true for P_Θ -a.a. $\theta \in \mathbb{R}^d$. The proof runs by induction on n . In fact fix on an arbitrary $t > 0$.

• $n = 0$. For any $\theta \notin L_1$ applying condition $(*)$ we get that

$$P_\theta(\{N_t = 0\}) = P_\theta(\{t < W_1\}) = 1 - F_\theta(t) > 0.$$

• Fix on arbitrary $t > 0$ and $\theta \notin L_1 \cup L_2$, and assume that for some $n \in \mathbb{N}_0$ condition $P_\theta(\{N_t = n\}) > 0$ holds true. It then follows that

$$\begin{aligned} P_\theta(\{N_t = n + 1\}) &= P_\theta(\{T_{n+1} \leq t < T_{n+2}\}) = P_\theta(\{T_{n+1} \leq t\}) - P_\theta(\{T_{n+2} \leq t\}) \\ &= F_\theta^{(n+1)*}(t) - F_\theta^{(n+2)*}(t) \\ &= \int_0^t F_\theta^{n*}(t-x) f_\theta(x) dx - \int_0^t F_\theta^{(n+1)*}(t-x) f_\theta(x) dx \end{aligned}$$

where $f_\theta(t) := F'_\theta(t)$, F_θ^{n*} is the n th convolution of F_θ , and the third equality follows by the fact that W is P_θ -i.i.d. according to step (a); hence

$$(1) \quad P_\theta(\{N_t = n + 1\}) = \int_0^t \left[F_\theta^{n*}(t-x) - F_\theta^{(n+1)*}(t-x) \right] f_\theta(x) dx.$$

Moreover, we have

$$\begin{aligned} P_\theta(\{N_t = n\}) > 0 &\iff P_\theta(\{T_n \leq t\}) > P_\theta(\{T_{n+1} \leq t\}) \\ &\iff F_\theta^{n*}(t) > F_\theta^{(n+1)*}(t) \end{aligned}$$

where the last equivalence follows by step (a). The latter together with condition $(*)$ yields that $\left[F_\theta^{n*}(t-x) - F_\theta^{(n+1)*}(t-x) \right] f_\theta(x) > 0$ for all $x \in (0, t)$, implying together with (1) that $P_\theta(\{N_t = n + 1\}) > 0$.

(c) For every $s, t \in \mathbb{R}_+$ with $s < t$ condition

c1

$$P(\{N_s = N_t = 1\}) = -\mathbb{E}_{P_\Theta} \left[\int_0^s G_\bullet(t-x) G'_\bullet(x) dx \right] > 0,$$

where $G_\theta(t) := 1 - F_\theta(t)$ for any $\theta \in \mathbb{R}^d$, holds true.

In fact, for every $s, t \in \mathbb{R}_+$ with $s < t$ we obtain

$$\begin{aligned}
P(\{N_s = N_t = 1\}) &= P(\{T_1 \leq s < t < T_2\}) \\
&= P(\{W_1 \leq s\}) - P(\{W_1 \leq s, W_2 \leq t - W_1\}) \\
&= \int_{\mathbb{R}^d \cap (L_1 \cup L_2)^c} \left[\int_0^s f_\theta(x) dx - \int_0^s \int_0^{t-x} f_\theta(y) f_\theta(x) dy dx \right] P_\Theta(d\theta) \\
&= -\mathbb{E}_{P_\Theta} \left[\int_0^s G_\bullet(t-x) G'_\bullet(x) dx \right],
\end{aligned}$$

where the third equality follows from [6], Lemma 3.5, and step (a). Assume now, if possible, that $P(\{N_s = N_t = 1\}) = 0$. Due to assumption (*), the latter is equivalent to the fact that for all $\theta \notin L_1 \cup L_2$ and $x \in (0, s]$ condition $G_\theta(t-x) = 0$ holds true; hence $f_\theta(t-x) = 0$ for every $x \in (0, s]$, a contradiction according to (*).

Note that, due to steps (b) and (c) all conditional probabilities considered in the next three steps are well defined.

(d) For any $0 < t, v$ condition

d1

$$(2) \quad \frac{\mathbb{E}_{P_\Theta} [-G_\bullet(t+v)p(\bullet)]}{\mathbb{E}_{P_\Theta} [G_\bullet(v)G'_\bullet(t)]} = \frac{\mathbb{E}_{P_\Theta} [-G_\bullet(t)p(\bullet)]}{\mathbb{E}_{P_\Theta} [G'_\bullet(t)]} = 1,$$

holds true.

For the proof of step (d) we follow arguments similar to those of [5] proof of Theorem 3.

In fact, for any $0 < u < t$ and $v > 0$ applying the Markov property we have

$$\begin{aligned}
P(\{N_{t-u} = N_t = N_{t+v} = 1\}) &= P(\{N_{t+v} = 1\} | \{N_t = 1\}) P(\{N_{t-u} = N_t = 1\}) \\
&\iff \frac{P(\{N_{t-u} = N_t = N_{t+v} = 1\})}{P(\{N_{t-u} = N_t = 1\})} = P(\{N_{t+v} = 1\} | \{N_t = 1\}) \\
&\iff \frac{P(\{N_{t-u} = N_{t+v} = 1\})}{P(\{N_{t-u} = N_t = 1\})} = \frac{A_1(t, v)}{B_1(t)}
\end{aligned}$$

where $A_1(t, v) := P(\{N_t = N_{t+v} = 1\})$ and $B_1(t) := P(\{N_t = 1\})$. Then, by step (c) we obtain that

$$P(\{N_{t-u} = N_{t+v} = 1\}) = -\mathbb{E}_{P_\Theta} \left[\int_0^{t-u} G_\bullet(t+v-x) G'_\bullet(x) dx \right]$$

and

$$P(\{N_{t-u} = N_t = 1\}) = -\mathbb{E}_{P_\Theta} \left[\int_0^{t-u} G_\bullet(t-x) G'_\bullet(x) dx \right].$$

Thus,

$$\frac{\mathbb{E}_{P_\Theta} \left[\int_0^{t-u} G_\bullet(t+v-x) G'_\bullet(x) dx \right]}{\mathbb{E}_{P_\Theta} \left[\int_0^{t-u} G_\bullet(t-x) G'_\bullet(x) dx \right]} = \frac{A_1(t, v)}{B_1(t)}.$$

It is obvious that the right side of the last equation is independent of u . Therefore its derivative with respect to u must be equal to zero; hence, equating the derivative of the left side with zero and applying the Dominated Convergence Theorem we have

$$\frac{\mathbb{E}_{P_\Theta} \left[\int_0^{t-u} G_\bullet(t+v-x) G'_\bullet(x) dx \right]}{\mathbb{E}_{P_\Theta} \left[\int_0^{t-u} G_\bullet(t-x) G'_\bullet(x) dx \right]} = \frac{\mathbb{E}_{P_\Theta} \left[G_\bullet(u+v) G'_\bullet(t-u) \right]}{\mathbb{E}_{P_\Theta} \left[G_\bullet(u) G'_\bullet(t-u) \right]}.$$

Since the left side of the last equality is independent of u the same must hold for the right one, so for $u \rightarrow 0$, $u \rightarrow t$ and by the Dominated Convergence Theorem we obtain

$$(3) \quad \frac{\mathbb{E}_{P_\Theta} \left[-G_\bullet(t+v) p(\bullet) \right]}{\mathbb{E}_{P_\Theta} \left[G_\bullet(v) G'_\bullet(t) \right]} = \frac{\mathbb{E}_{P_\Theta} \left[-G_\bullet(t) p(\bullet) \right]}{\mathbb{E}_{P_\Theta} \left[G'_\bullet(t) \right]}.$$

Moreover, since the right side of the (3) is independent of v its derivative with respect to v must be equal to zero. Thus, equating the derivative of the left side with zero and applying the Dominated Convergence Theorem we have

$$\frac{\mathbb{E}_{P_\Theta} \left[-G_\bullet(t+v) p(\bullet) \right]}{\mathbb{E}_{P_\Theta} \left[G_\bullet(v) G'_\bullet(t) \right]} = \frac{\mathbb{E}_{P_\Theta} \left[-G'_\bullet(t+v) p(\bullet) \right]}{\mathbb{E}_{P_\Theta} \left[G'_\bullet(v) G'_\bullet(t) \right]}.$$

Thus, for $v \rightarrow 0$ and by the Dominated Convergence Theorem we obtain that

$$\frac{\mathbb{E}_{P_\Theta} \left[-G_\bullet(t) p(\bullet) \right]}{\mathbb{E}_{P_\Theta} \left[G'_\bullet(t) \right]} = \frac{\mathbb{E}_{P_\Theta} \left[-G'_\bullet(t) p(\bullet) \right]}{\mathbb{E}_{P_\Theta} \left[-p(\bullet) G'_\bullet(t) \right]} = 1;$$

implying together with condition (3) that

$$\frac{\mathbb{E}_{P_\Theta} \left[-G_\bullet(t+v) p(\bullet) \right]}{\mathbb{E}_{P_\Theta} \left[G_\bullet(v) G'_\bullet(t) \right]} = \frac{\mathbb{E}_{P_\Theta} \left[-G_\bullet(t) p(\bullet) \right]}{\mathbb{E}_{P_\Theta} \left[G'_\bullet(t) \right]} = 1.$$

(e) For any $0 < t, v$ condition

e1

$$(4) \quad \mathbb{E}_{P_\Theta} \left[G_\bullet(t+v) (p(\bullet))^2 \right] = \mathbb{E}_{P_\Theta} \left[-G_\bullet(v) G'_\bullet(t) p(\bullet) \right]$$

holds true.

In fact, for any $0 < u < t$, $v > 0$ and $w > 0$ applying the Markov property we have

$$\begin{aligned} & P(\{N_{t-u} = N_t = 1, N_{t+v} = N_{t+v+w} = 2\}) \\ &= P(\{N_{t+v+w} = 2\} | \{N_{t+v} = 2\}) P(\{N_{t+v} = 2\} | \{N_t = 1\}) P(\{N_{t-u} = N_t = 1\}) \end{aligned}$$

or equivalently

$$\frac{P(\{N_{t-u} = N_t = 1, N_{t+v} = N_{t+v+w} = 2\})}{P(\{N_{t-u} = N_t = 1\})} = \frac{A_2(t, v, w)}{B_2(t, v)},$$

where $A_2(t, v, w) = P(\{N_{t+v+w} = 2, N_{t+v} = 2\}) \cdot P(\{N_{t+v} = 2, N_t = 1\})$ and $B_2(t, v) = P(\{N_{t+v} = 2\}) \cdot P(\{N_t = 1\})$.

Moreover working as in the proof of (c) we get

$$\begin{aligned} & P(\{N_{t-u} = N_t = 1, N_{t+v} = N_{t+v+w} = 2\}) \\ &= \mathbb{E}_{P_\Theta} \left[\int_0^{t-u} \int_{t-x}^{t+v-x} G_\bullet(t+v+w-x-y) G'_\bullet(y) G'_\bullet(x) dy dx \right], \end{aligned}$$

implying together with (c) that

$$\frac{\mathbb{E}_{P_\Theta} \left[\int_0^{t-u} \int_{t-x}^{t+v-x} G_\bullet(t+v+w-x-y) G'_\bullet(y) G'_\bullet(x) dy dx \right]}{\mathbb{E}_{P_\Theta} \left[\int_0^{t-u} G_\bullet(t-x) G'_\bullet(x) dx \right]} = -\frac{A_2(t, v, w)}{B_2(t, v)}.$$

It is obvious that the right side of the last equation is independent of u . Therefore its derivative with respect to u must be equal to zero. Furthermore, equating the left's side derivative with zero and applying the Dominated Convergence Theorem we have

$$\begin{aligned} & \frac{\mathbb{E}_{P_\Theta} \left[\int_0^{t-u} \int_{t-x}^{t+v-x} G_\bullet(t+v+w-x-y) G'_\bullet(y) G'_\bullet(x) dy dx \right]}{\mathbb{E}_{P_\Theta} \left[\int_0^{t-u} G_\bullet(t-x) G'_\bullet(x) dx \right]} \\ &= \frac{\mathbb{E}_{P_\Theta} \left[\int_u^{u+v} G_\bullet(u+v+w-y) G'_\bullet(y) G'_\bullet(t-u) dy \right]}{\mathbb{E}_{P_\Theta} [G_\bullet(u) G'_\bullet(t-u)]}. \end{aligned}$$

Since the left side of the above equality is independent of u the same must hold for the right one, so for $u \rightarrow 0$, $u \rightarrow t$ and by the Dominated Convergence Theorem we obtain

$$\frac{\mathbb{E}_{P_\Theta} \left[-\int_t^{t+v} G_\bullet(t+v+w-y) G'_\bullet(y) p(\bullet) dy \right]}{\mathbb{E}_{P_\Theta} \left[\int_0^v G_\bullet(v+w-y) G'_\bullet(y) G'_\bullet(t) dy \right]} = \frac{\mathbb{E}_{P_\Theta} [-G_\bullet(t) p(\bullet)]}{\mathbb{E}_{P_\Theta} [G'_\bullet(t)]} = 1,$$

where the last equality can be rewritten in the following form

$$\mathbb{E}_{P_\Theta} \left[-\int_t^{t+v} G_\bullet(t+v+w-y) G'_\bullet(y) p(\bullet) dy \right] = \mathbb{E}_{P_\Theta} \left[\int_0^v G_\bullet(v+w-y) G'_\bullet(y) G'_\bullet(t) dy \right]$$

due to step (d). If we take the derivative with respect to v and we apply the Dominated Convergence Theorem in the above equality we obtain that

$$\mathbb{E}_{P_\Theta} [-G_\bullet(w) G'_\bullet(t+v) p(\bullet)] = \mathbb{E}_{P_\Theta} [G_\bullet(w) G'_\bullet(v) G'_\bullet(t)].$$

Derivating now with respect to w and applying once again the Dominated Convergence Theorem we get that

$$\mathbb{E}_{P_\Theta} [-G'_\bullet(w)G'_\bullet(t+v)p(\bullet)] = \mathbb{E}_{P_\Theta} [G'_\bullet(w)G'_\bullet(v)G'_\bullet(t)].$$

By letting now $w \rightarrow 0$ and by the Dominated Convergence Theorem we obtain that

$$\mathbb{E}_{P_\Theta} [G'_\bullet(t+v)(p(\bullet))^2] = \mathbb{E}_{P_\Theta} [-G'_\bullet(v)G'_\bullet(t)p(\bullet)].$$

Integration with respect to v yields

$$\mathbb{E}_{P_\Theta} [G_\bullet(t+v)(p(\bullet))^2] = \mathbb{E}_{P_\Theta} [-G_\bullet(v)G'_\bullet(t)p(\bullet)].$$

(f) For any $0 < t, v$ condition

f1

$$(5) \quad \mathbb{E}_{P_\Theta} [G_\bullet(t)G_\bullet(v)(p(\bullet))^2] = \mathbb{E}_{P_\Theta} [-G_\bullet(v)G'_\bullet(t)p(\bullet)]$$

holds true.

In fact, for any $0 < u < t$, $v > 0$ and $w > 0$ by the Markov property we have

$$\begin{aligned} & P(\{N_{t-u} = N_t = 1, N_{t+v} = 2, N_{t+v+w} = 3\}) \\ &= P(\{N_{t+v+w} = 3\} | \{N_{t+v} = 2\}) P(\{N_{t+v} = 2\} | \{N_t = 1\}) P(\{N_{t-u} = N_t = 1\}), \end{aligned}$$

or equivalently

$$\frac{P(\{N_{t-u} = N_t = 1, N_{t+v} = 2, N_{t+v+w} = 3\})}{P(\{N_{t-u} = N_t = 1\})} = \frac{A_3(t, v, w)}{B_3(t, v)},$$

where $A_3(t, v, w) = P(\{N_{t+v+w} = 3, N_{t+v} = 2\}) \cdot P(\{N_{t+v} = 2, N_t = 1\})$ and $B_3(t, v) = P(\{N_{t+v} = 2\}) \cdot P(\{N_t = 1\})$.

Moreover, applying [6], Lemma 3.5, of [6] and step (a) after some manipulation as in the proof of (c) we obtain

$$\begin{aligned} & P(\{N_{t-u} = N_t = 1, N_{t+v} = 2, N_{t+v+w} = 3\}) \\ &= \mathbb{E}_{P_\Theta} \left[\int_0^{t-u} \int_{t-x}^{t+v-x} \int_{t+v-x-y}^{t+v+w-x-y} G_\bullet(t+v+w-x-y-z) f_\bullet(z) f_\bullet(y) f_\bullet(x) dz dy dx \right]. \end{aligned}$$

The latter together with step (c) yields

$$\begin{aligned} & \frac{\mathbb{E}_{P_\Theta} \left[\int_0^{t-u} \int_{t-x}^{t+v-x} \int_{t+v-x-y}^{t+v+w-x-y} G_\bullet(t+v+w-x-y-z) G'_\bullet(z) G'_\bullet(y) G'_\bullet(x) dz dy dx \right]}{\mathbb{E}_{P_\Theta} \left[\int_0^{t-u} G_\bullet(t-x) G'_\bullet(x) dx \right]} \\ &= \frac{A_3(t, v, w)}{B_3(t, v)}. \end{aligned}$$

It is obvious that the right side of the last equation is independent of u . Therefore its derivative with respect to u must be equal to zero; hence, equating the derivative of the left side with zero and applying the Dominated Convergence Theorem we have

$$\begin{aligned} & \frac{\mathbb{E}_{P_\Theta} \left[\int_0^{t-u} \int_{t-x}^{t+v-x} \int_{t+v-x-y}^{t+v+w-x-y} G_\bullet(t+v+w-x-y-z) f_\bullet(z) f_\bullet(y) f_\bullet(x) dz dy dx \right]}{\mathbb{E}_{P_\Theta} \left[\int_0^{t-u} G_\bullet(t-x) G'_\bullet(x) dx \right]} \\ &= \frac{\mathbb{E}_{P_\Theta} \left[\int_u^{u+v} \int_{u+v-y}^{u+v+w-y} G_\bullet(u+v+w-y-z) G'_\bullet(z) G'_\bullet(y) G'_\bullet(t-u) dz dy \right]}{\mathbb{E}_{P_\Theta} [G_\bullet(u) G'_\bullet(t-u)]}. \end{aligned}$$

Since the left side of the last equality is independent of u the same must hold for the right one, so for $u \rightarrow 0$ and $u \rightarrow t$ we obtain

$$\frac{\mathbb{E}_{P_\Theta} \left[- \int_t^{t+v} \int_{t+v-y}^{t+v+w-y} G_\bullet(t+v+w-y-z) G'_\bullet(z) G'_\bullet(y) p(\bullet) dz dy \right]}{\mathbb{E}_{P_\Theta} \left[\int_0^v \int_{v-y}^{v+w-y} G_\bullet(v+w-y-z) G'_\bullet(z) G'_\bullet(y) G'_\bullet(t) dz dy \right]} = \frac{\mathbb{E}_{P_\Theta} [-G_\bullet(t) p(\bullet)]}{\mathbb{E}_{P_\Theta} [G'_\bullet(t)]}$$

Due to condition (2), the last equality can be rewritten in the following form

$$\begin{aligned} & \mathbb{E}_{P_\Theta} \left[\int_0^v \int_{v-y}^{v+w-y} G_\bullet(v+w-y-z) G'_\bullet(z) G'_\bullet(y) G'_\bullet(t) dz dy \right] \\ &= \mathbb{E}_{P_\Theta} \left[- \int_t^{t+v} \int_{t+v-y}^{t+v+w-y} G_\bullet(t+v+w-y-z) G'_\bullet(z) G'_\bullet(y) p(\bullet) dz dy \right]. \end{aligned}$$

If we take the derivative with respect to v and we apply the Dominated Convergence Theorem in the above equality we obtain that

$$\begin{aligned} & \mathbb{E}_{P_\Theta} \left[\int_0^w G_\bullet(w-z) G'_\bullet(z) G'_\bullet(v) G'_\bullet(t) dz \right] \\ &= \mathbb{E}_{P_\Theta} \left[- \int_0^w G_\bullet(w-z) G'_\bullet(z) G'_\bullet(t+v) p(\bullet) dz \right], \end{aligned}$$

or equivalently if we put $s := w - z$ in the first integral we obtain

$$\begin{aligned} & \mathbb{E}_{P_\Theta} \left[\int_0^w G_\bullet(s) G'_\bullet(w-s) G'_\bullet(v) G'_\bullet(t) ds \right] \\ &= \mathbb{E}_{P_\Theta} \left[- \int_0^w G_\bullet(w-z) G'_\bullet(z) G'_\bullet(t+v) p(\bullet) dz \right]. \end{aligned}$$

Derivating now with respect to w and applying once again the Dominated Convergence Theorem we get that

$$\mathbb{E}_{P_\Theta} [-G_\bullet(w) p(\bullet) G'_\bullet(v) G'_\bullet(t)] = \mathbb{E}_{P_\Theta} [-G'_\bullet(w) p(\bullet) G'_\bullet(t+v)].$$

Integration with respect to t yields

$$\mathbb{E}_{P_\Theta} [-G_\bullet(t)G_\bullet(w)p(\bullet)G'_\bullet(v)] = \mathbb{E}_{P_\Theta} [-G_\bullet(t+v)G'_\bullet(w)p(\bullet)].$$

For $v \rightarrow 0$ any by the Dominated Convergence Theorem we get

$$\mathbb{E}_{P_\Theta} [G_\bullet(t)G_\bullet(w)(p(\bullet))^2] = \mathbb{E}_{P_\Theta} [-G_\bullet(t)G'_\bullet(w)p(\bullet)].$$

Take now $v \rightarrow t$, the by the Dominated Convergence Theorem we get that

$$\mathbb{E}_{P_\Theta} [G_\bullet(v)G_\bullet(w)(p(\bullet))^2] = \mathbb{E}_{P_\Theta} [-G_\bullet(v)G'_\bullet(w)p(\bullet)].$$

Finally by letting $w \rightarrow t$ and by the Dominated Convergence Theorem we obtain

$$\mathbb{E}_{P_\Theta} [G_\bullet(t)G_\bullet(v)(p(\bullet))^2] = \mathbb{E}_{P_\Theta} [-G_\bullet(v)G'_\bullet(t)p(\bullet)].$$

(g) For every $t > 0$ condition

g1

$$(6) \quad \mathbb{E}_{P_\Theta} \left[(G'_\bullet(t) + p(\bullet)G_\bullet(t))^2 \right] = 0$$

holds true.

In fact, let us fix on arbitrary $t > 0$. Applying (4) and (5) for $v \rightarrow t$ we get

$$\begin{aligned} & \mathbb{E}_{P_\Theta} \left[(G'_\bullet(t) + p(\bullet)G_\bullet(t))^2 \right] \\ &= \mathbb{E}_{P_\Theta} [G_\bullet'^2(t) - G_\bullet^2(t)p^2(\bullet) - G_\bullet(2t)p^2(\bullet) + p^2(\bullet)G_\bullet^2(t)]; \end{aligned}$$

hence

$$(7) \quad \mathbb{E}_{P_\Theta} \left[(G'_\bullet(t) + p(\bullet)G_\bullet(t))^2 \right] = \mathbb{E}_{P_\Theta} [G_\bullet'^2(t) - G_\bullet(2t)p^2(\bullet)].$$

Derivating (2) with respect to v and applying the Dominated Convergence Theorem we get

$$\mathbb{E}_{P_\Theta} [-G'_\bullet(t+v)h(\bullet)] = \mathbb{E}_{P_\Theta} [G'_\bullet(v)G'_\bullet(t)];$$

hence by letting $v \rightarrow t$ we obtain

$$\mathbb{E}_{P_\Theta} [-G'_\bullet(2t)p(\bullet)] = \mathbb{E}_{P_\Theta} [G_\bullet'^2(t)].$$

Equation (7) can be rewritten as

$$(8) \quad \mathbb{E}_{P_\Theta} \left[(G'_\bullet(t) + p(\bullet)G_\bullet(t))^2 \right] = \mathbb{E}_{P_\Theta} [-G'_\bullet(2t)p(\bullet) - G_\bullet(2t)p^2(\bullet)].$$

Taking now $v \rightarrow 0$ in equation (4) we obtain that

$$\mathbb{E}_{P_\Theta} [G_\bullet(t)p^2(\bullet)] = \mathbb{E}_{P_\Theta} [-G'_\bullet(t)p(\bullet)];$$

hence substituting t by $2t$ we get that

$$(9) \quad \mathbb{E}_{P_\Theta} [G_\bullet(2t)p^2(\bullet)] = \mathbb{E}_{P_\Theta} [-G'_\bullet(2t)p(\bullet)].$$

Thus, by equations (8) and (9) it follows that

$$\mathbb{E}_{P_\Theta} \left[\left(G'_\bullet(t) + p(\bullet)G_\bullet(t) \right)^2 \right] = 0.$$

(h) There exists a P_Θ -null set $L_3 \in \mathfrak{B}_d$ such that for any $\theta \notin L_3$ the process W is P_θ -exponentially distributed with parameter $p(\theta)$. h1

In fact, step (g) yields that for any $t > 0$ there exists a P_Θ -null set $M_t \in \mathfrak{B}_d$ such that for any $\theta \notin M_t$ condition

$$(10) \quad G'_\theta(t) + p(\theta)G_\theta(t) = 0$$

holds true. Put $L_3 := \bigcup_{s \in \mathbb{Q}_+} M_s$ and let $t > 0$ and $\theta \notin L_3$ be arbitrary. There exists a sequence $\{s_n\}_{n \in \mathbb{N}}$ in \mathbb{Q}_+ such that $t = \lim_{n \rightarrow \infty} s_n$; hence applying (10) we get

$$G'_\theta(s_n) = -p(\theta)G_\theta(s_n) \quad \forall n \in \mathbb{N}.$$

But then applying assumption (*) we obtain

$$G'_\theta(t) = \lim_{n \rightarrow \infty} G'_\theta(s_n) = -p(\theta)G_\theta(t);$$

hence $G_\theta(t) = e^{-p(\theta)t}$, implying that $(P_\theta)_{W_n} = \mathbf{Exp}(p(\theta))$ for all $n \in \mathbb{N}$.

(i) Put $L_* := L_1 \cup L_2 \cup L_3$ and denote again by p the restriction of p to L_*^c . Define $\check{\Theta} : \Omega \longrightarrow (0, \infty)$ by $\check{\Theta} := \check{p} \circ \Theta$, where i1

$$\check{p}(\theta) := \begin{cases} p(\theta) & \text{if } \theta \notin L_* \\ 1 & \text{otherwise.} \end{cases}$$

Define $r := p^{-1} : p(L_*^c) \longrightarrow L_*^c$, and $\hat{r} : (0, \infty) \longrightarrow \mathbb{R}^d$ as well as the family $\{Q_{\check{\theta}}\}_{\check{\theta} > 0}$ as in Lemma 2.4 (for $k = 1$, and p , r and \hat{r} in place of h , g and \hat{g} , respectively). Then the family $\{Q_{\check{\theta}}\}_{\check{\theta} > 0}$ is a disintegration of P over $P_{\check{\Theta}}$ consistent with $\check{\Theta}$, and for any $\check{\theta} \notin M$, where $M := p(L_*)$ is defined as in Lemma 2.4, the random variable W_n is $Q_{\check{\theta}}$ -exponentially distributed with parameter $\check{\theta}$, and W is $Q_{\check{\theta}}$ -independent.

In fact, by Lemma 2.4 the family $\{Q_{\check{\theta}}\}_{\check{\theta} > 0}$ is a disintegration of P over $P_{\check{\Theta}}$ consistent with $\check{\Theta}$, while for any $\check{\theta} \notin M$ again by Lemma 2.4, and the steps (a) and (h) follows that the process W is $Q_{\check{\theta}}$ -independent and each W_n is $Q_{\check{\theta}}$ -exponentially distributed with parameter $\check{\theta}$ respectively.

(j) N is a P -MPP($\check{\Theta}$). j1

In fact, by step (i) we obtain that for any $\check{\theta} \notin L_*$ the counting process N is $Q_{\check{\theta}}$ -PP($\check{\Theta}$) (cf. e.g [10], Theorem 2.3.4). Thus, applying [6], Proposition 4.4, we deduce that N is a P -MPP($\check{\Theta}$). □

Remarks 2.8 (a) Assumptions (*) and (**) trace back to Huang [5], Theorem 3, and is essential in Grandell's study of MRPs (see Grandell [4], Section 5.3.)

(b) We inserted the proof of step (f) of Proposition 2.7 in a different way than that of Huang's [5], proof of condition (16) of Theorem 3, since we could not prove it as suggested by Huang. In fact, starting from the probability $P(\{N_{t-u} = N_t = 1, N_{t+v} = N_{t+v+w} = 2\})$, as suggested by Huang, we could prove only condition (15) instead of condition (16) of Huang [5] as it is shown in step (e) of the proof of Proposition 2.7.

(c) In condition (*) Huang assumed the stronger condition $0 < F'_\theta(t) < C$ for any $t > 0$ and $\theta \notin L_1$, where C is a positive constant, in the place of $0 < F'_\theta(t) < C(\theta)$ for any $t > 0$ and $\theta \notin L_1$ for $C \in \mathcal{L}^1(P_\Theta)$ in Proposition 2.7.

The following result extends Lemma 3.4 from [8].

Lemma 2.9 *Let $\{P_\theta\}_{\theta \in \mathbb{R}^d}$ be a disintegration of P over P_Θ consistent with Θ , let $\{k_n\}_{n \in \mathbb{N}}$ be a sequence of T - \mathfrak{B}_d -Markov kernels, let h be a \mathbb{R}^k -valued (for $k \in \mathbb{N}$) \mathfrak{B}_d - T -measurable map on \mathbb{R}^d , and let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of Σ - T -measurable maps from Ω into \mathcal{T} . Suppose that T is countably generated. Then the following are equivalent:*

$$(i) \exists L_4 \in \sigma(\Theta)_0 \quad \forall n \in \mathbb{N} \quad P_{X_n|\Theta} \upharpoonright T \times L_4^c = \mathbf{K}_n(h(\Theta)) \upharpoonright T \times L_4^c;$$

$$(ii) \exists \tilde{L}_4 \in (\mathfrak{B}_d)_0 \quad \forall n \in \mathbb{N} \quad \forall \theta \notin \tilde{L}_4 \quad (P_\theta)_{X_n} = \mathbf{K}_n(h(\theta)),$$

where $\sigma(\Theta)_0 := \{M \in \sigma(\Theta) : P(M) = 0\}$ and $(\mathfrak{B}_d)_0 := \{\tilde{M} \in \mathfrak{B}_d : P_\Theta(\tilde{M}) = 0\}$.

Proof. Ad $(i) \implies (ii)$. Assume that there exists a set $L_4 \in \sigma(\Theta)_0$ such that for every $n \in \mathbb{N}$ condition

$$P_{X_n|\Theta} \upharpoonright T \times L_4^c = \mathbf{K}_n(h(\Theta)) \upharpoonright T \times L_4^c$$

holds true. Then for any fixed $n \in \mathbb{N}$, $D \in \mathfrak{B}_d$ and $B \in T$ we obtain

$$\int_{\Theta^{-1}(D)} P_{X_n|\Theta}(B, \bullet) dP = \int_{\Theta^{-1}(D)} \mathbf{K}_n(h(\Theta))(B, \bullet) dP;$$

hence taking into account [6] Lemma 3.5 we get

$$\int_D (P_\theta)_{X_n}(B) P_\Theta(d\theta) = \int_D \mathbf{K}_n(h(\theta))(B) P_\Theta(d\theta)$$

Consequently, there exists a set $\tilde{L}_{n,B} \in (\mathfrak{B}_d)_0$ such that

$$(11) \quad (P_\theta)_{X_n}(B) = \mathbf{K}_n(h(\theta))(B) \quad \forall \theta \notin \tilde{L}_{n,B}.$$

Put $\tilde{L}_4 := \bigcup_{n \in \mathbb{N}} \bigcup_{B \in \mathcal{G}_T} \tilde{L}_{n,B}$, where \mathcal{G}_T is a countable generator of T being closed under finite intersections, and denote by \mathcal{D} the class of all $B \in T$ such that condition (11) is

satisfied for every $\theta \notin \tilde{L}_4$ and $n \in \mathbb{N}$. It can be easily seen that $\mathcal{G}_T \subseteq \mathcal{D}$ and that \mathcal{D} is a Dynkin class, implying that $\mathcal{D} = T$. Thus assertion (ii) follows.

Applying a similar reasoning we obtain the converse implication. \square

The following result shows how to reduce an extended MRP with parameters Θ and h to a $(P, \mathbf{K}(\tilde{\Theta}))$ -MRP under the change of the parameter.

lem3

Lemma 2.10 *Let h and $\tilde{\Theta}$ be as in Lemma 2.4. Suppose that N is an extended MRP on (Ω, Σ, P) with parameters Θ and h . Then N is a $(P, \mathbf{K}(\tilde{\Theta}))$ -MRP.*

Proof. Let $\{P_\theta\}_{\theta \in \mathbb{R}^d}$, g and $\{Q_{\tilde{\theta}}\}_{\tilde{\theta} \in \mathbb{R}^k}$ be as in Lemma 2.4. According to Lemma 2.9, there exists a set $\tilde{L}_4 \in (\mathfrak{B}_d)_0$ such that $(P_\theta)_{W_n} = \mathbf{K}(h(\theta))$ for all $\theta \notin \tilde{L}_4$. We may and do assume that \tilde{L}_4 contains the null set L of Lemma 2.4. Applying now Lemma 2.4, we obtain that $\{Q_{\tilde{\theta}}\}_{\tilde{\theta} \in \mathbb{R}^k}$ is a disintegration of P over $P_{\tilde{\Theta}}$ consistent with $\tilde{\Theta}$, and for all $\tilde{\theta} \notin \tilde{M}_4 := g^{-1}(\tilde{L}_4)$ the process W is $Q_{\tilde{\theta}}$ -independent and $(Q_{\tilde{\theta}})_{W_n} = \mathbf{K}(\tilde{\theta})$ for every $n \in \mathbb{N}$. But the latter together with [8] Proposition 3.8 yields the conclusion of the lemma. \square

The next result extends Proposition 2.7.

thm1

Theorem 2.11 *Let $\{P_\theta\}_{\theta \in \mathbb{R}^d}$ be a disintegration of P over P_Θ consistent with Θ , let h, L be as in Lemma 2.4, and let N be an extended MRP on (Ω, Σ, P) with parameters Θ and h . For any $\theta \in \mathbb{R}^d$ and $t \in \mathbb{R}_+$ put $F_{h(\theta)}(t) := P_\theta(\{W_n \leq t\})$ for all $n \in \mathbb{N}$. Assume that there exists a P_Θ -null set $O_1 \in \mathfrak{B}_d$ such that for any $\theta \notin O_1$ the function $F_{h(\theta)}$ satisfies condition (*), and the function $q := q_h : O_1^c \rightarrow \mathbb{R}$ defined by means of $q(\theta) := \lim_{t \rightarrow 0} F'_{h(\theta)}(t)$ satisfies condition (**).*

Then the following assertions are equivalent:

- (i) N has the multinomial property;
- (ii) N has the Markov property;
- (iii) there exists a random variable $\hat{\Theta}$ on Ω such that N is a $MPP(\hat{\Theta})$.

Proof. Let $\tilde{\Theta}$ and $\{Q_{\tilde{\theta}}\}_{\tilde{\theta} \in \mathbb{R}^k}$ be as in Lemma 2.4. It then follows by Lemma 2.4 together with Lemma 2.9 that the process N is a $(P, \mathbf{K}(\tilde{\Theta}))$ -MRP. For any $\tilde{\theta} \in \mathbb{R}^k$ and $t \in \mathbb{R}_+$ put $F_{\tilde{\theta}}(t) := Q_{\tilde{\theta}}(\{W_n \leq t\})$ for all $n \in \mathbb{N}$.

(a) There exists a $P_{\tilde{\Theta}}$ -null set \tilde{V}_1 in \mathfrak{B}_k such that for any $\tilde{\theta} \notin \tilde{V}_1$ the function $F_{\tilde{\theta}}$ is continuously differentiable on $(0, \infty)$ and $0 < F'_{\tilde{\theta}}(t) < C(\theta)$ for each $t > 0$.

In fact, since $P_{W_n|\Theta} = \mathbf{K}(h(\Theta))$ $P \upharpoonright \sigma(\Theta)$ -a.s. for any $n \in \mathbb{N}$, it follows by Lemma 2.9 that there exists a P_Θ -null set $\tilde{L}_4 \in \mathfrak{B}_d$ such that for any $n \in \mathbb{N}$ and $\theta \notin \tilde{L}_4$ condition $(P_\theta)_{W_n} = \mathbf{K}(h(\theta))$ holds true. On the other hand, according to condition (*) there exists a P_Θ -null set $O_1 \in \mathfrak{B}_d$ such that for each $\theta \notin O_1$ the function $F_{h(\theta)}$ is continuously differentiable on $(0, \infty)$ and $0 < F'_{h(\theta)}(t) < C(\theta)$. Put $\tilde{O}_1 := \tilde{L}_4 \cup O_1$.

By Lemma 2.4 the family $\{Q_{\tilde{\theta}}\}_{\tilde{\theta} \in \mathbb{R}^k}$ is a disintegration of P over $P_{\tilde{\Theta}}$ consistent with $\tilde{\Theta}$ such that for any $\tilde{\theta} \notin \tilde{V}_1 := h(\tilde{O}_1) \in \mathfrak{B}_k$ the process W is $Q_{\tilde{\theta}}$ -independent and $(Q_{\tilde{\theta}})_{W_n} = \mathbf{K}(\tilde{\theta})$ for any $n \in \mathbb{N}$. Then for any $\tilde{\theta} \notin \tilde{V}_1$ there exists exactly one $\theta \in \tilde{O}_1$ such that $\tilde{\theta} = h(\theta)$ and for any $t \in \mathbb{R}_+$

$$(12) \quad F_{\tilde{\theta}}(t) = \mathbf{K}(\tilde{\theta})((-\infty, t]) = \mathbf{K}(h(\theta))((-\infty, t]) = F_{h(\theta)}(t),$$

implying that condition (a) holds true.

(b) For any $\tilde{\theta} \notin \tilde{V}_1$ there exists exactly one $\tilde{s}(\tilde{\theta}) := \lim_{t \rightarrow 0} F'_{\tilde{\theta}}(t) > 0$ and \tilde{s} is injective.

In fact, for any $\tilde{\theta} \notin \tilde{V}_1$ there exists exactly one $\theta \notin \tilde{O}_1$ with $\tilde{\theta} = h(\theta)$. It then follows by (**) and (12) that

$$\lim_{t \rightarrow 0} F'_{\tilde{\theta}}(t) = \lim_{t \rightarrow 0} F'_{h(\theta)}(t) = q(\theta) > 0;$$

hence \tilde{s} is positive and injective. Thus, (b) is valid.

Define $s : \mathbb{R}^k \rightarrow \mathbb{R}$ by

$$s(\tilde{\theta}) := \begin{cases} \tilde{s}(\tilde{\theta}) & \text{if } \tilde{\theta} \notin \tilde{V}_1 \\ 1 & \text{otherwise} \end{cases}$$

and $\check{q} : \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$\check{q}(\theta) := \begin{cases} q(\theta) & \text{if } \theta \notin \tilde{O}_1 \\ 1 & \text{otherwise.} \end{cases}$$

Put $\hat{\Theta} := s \circ \tilde{\Theta} = s \circ h \circ \Theta = \check{q} \circ \Theta$. Since (a) and (b) are satisfied by W and $\{Q_{\tilde{\theta}}\}_{\tilde{\theta} \in \mathbb{R}^k}$ we may apply Proposition 2.7 for $\tilde{\Theta}$ and \tilde{s} in the place of Θ and p , respectively, to get the thesis of the theorem. \square

The following result may be of independent interest, since it ensures the permanence of the Markov and the multinomial property with respect to P to that with respect to the disintegrating measures $Q_{\hat{\theta}}$.

cor1

Corollary 2.12 *Let $\{P_{\theta}\}_{\theta \in \mathbb{R}^d}$, h and N be as in Theorem 2.11. Suppose that conditions (*) and (**) are satisfied. Then there exists a random variable $\hat{\Theta} = \check{q} \circ \Theta$ on Ω such that the following assertions are equivalent:*

(i) N has the P -Markov property;

(ii) N has the $Q_{\hat{\theta}}$ -Markov property for $P_{\hat{\Theta}} - a.a.$ $\hat{\theta} > 0$.

(iii) N has the $Q_{\hat{\theta}}$ -multinomial property for $P_{\hat{\Theta}} - a.a.$ $\hat{\theta} > 0$.

Proof. Ad (i) \implies (ii): Assume that assertion (i) holds true. Since N is an extended MRP with parameters Θ and h , it follows by Theorem 2.11 that N is a P -MPP($\hat{\Theta}$), where $\hat{\Theta} = \check{q} \circ \Theta$. Fix on an arbitrary $A \in \Sigma$ and put

$$Q_{\hat{\theta}}(A) := \begin{cases} (P_{\bullet}(A) \circ q^{-1})(\hat{\theta}) & \text{if } \hat{\theta} \notin q(\tilde{O}_1) \\ P(A) & \text{otherwise.} \end{cases}$$

By Lemma 2.4 the family $\{Q_{\hat{\theta}}\}_{\hat{\theta}>0}$ is a disintegration of P over $P_{\hat{\Theta}}$ consistent with $\hat{\Theta}$. Then according to [6], Proposition 4.2, N is a $Q_{\hat{\theta}}$ -PP($\hat{\theta}$) for $P_{\hat{\Theta}}$ -a.a. $\hat{\theta} > 0$ and thus it has the $Q_{\hat{\theta}}$ -Markov property.

Ad (ii) \implies (i): Assume that assertion (ii) holds true. Since N is an extended MRP with parameters Θ and h , it follows by Lemma 2.10 that N is a $(P, \mathbf{K}(\tilde{\Theta}))$ -MRP. Fix on an arbitrary $A \in \Sigma$ and put

$$Q_{\tilde{\theta}}(A) := \begin{cases} (P_{\bullet}(A) \circ h^{-1})(\tilde{\theta}) & \text{if } \tilde{\theta} \notin \tilde{V}_1 \\ P(A) & \text{otherwise,} \end{cases}$$

where $\tilde{\theta} = h(\theta)$. By Lemma 2.4 the family $\{Q_{\tilde{\theta}}\}_{\tilde{\theta}>0}$ is a disintegration of P over $P_{\tilde{\Theta}}$ consistent with $\tilde{\Theta}$. Applying now [8], Proposition 3.8, we obtain that N is a $(Q_{\tilde{\theta}}, \mathbf{K}(\tilde{\theta}))$ -RP. It follows easily that

$$Q_{\hat{\theta}}(A) = \begin{cases} Q_{\tilde{\theta}}(A) & \text{if } \hat{\theta} \notin s(\tilde{V}_1) \\ P(A) & \text{otherwise,} \end{cases}$$

where $\hat{\theta} = s(\tilde{\theta})$. Again by Lemma 2.4 the family $\{Q_{\hat{\theta}}\}_{\hat{\theta}>0}$ is a disintegration of P over $P_{\hat{\Theta}}$ consistent with $\hat{\Theta}$ and the process W is $Q_{\hat{\theta}}$ -independent. Moreover, for any $\hat{\theta} \notin s(\tilde{V}_1)$ we obtain $Q_{\hat{\theta}}(A) = Q_{\tilde{\theta}}(A) = \mathbf{K}(\tilde{\theta})$. As a consequence we get that N is a $(Q_{\hat{\theta}}, \mathbf{K}(\tilde{\theta}))$ -RP, implying together with Remark 2.6 that N is a $Q_{\hat{\theta}}$ -PP($\hat{\theta}$), where $\hat{\theta} = s(\tilde{\theta})$, for any $\hat{\theta} \notin s(\tilde{V}_1)$. Thus, taking into account [6], Proposition 4.2, we get that N is a P -MPP($\hat{\Theta}$); hence N has the P -Markov property (cf. e.g. [10] Theorem 4.2.3). Ad (ii) \implies (iii): Assume that (ii) holds true. Since (i) \iff (ii), it follows from Theorem 2.11, that (i) is equivalent to the fact that N is a P -MPP($\hat{\Theta}$). Applying now [6], Proposition 4.2, we obtain that N is a $Q_{\hat{\theta}}$ -PP($\hat{\theta}$) for $P_{\hat{\Theta}}$ -a.a. $\hat{\theta} > 0$. But the latter implies (iii).

Ad (iii) \implies (ii): The implication (iii) \implies (ii) follows by [13], Lemma 2.2.7. \square

r2

Remark 2.13 Assumptions (*) and (**) in Proposition 2.7 and Theorem 2.11 are essential.

In fact, consider the trivial counting process N defined by means of $N_t := [t]$ for every $t \in \mathbb{R}_+$. It can be easily proven that N is a Markov $(P, \mathbf{K}(\theta_0))$ -RP with

$$\mathbf{K}(\theta_0)(t) := \begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } t \geq 1. \end{cases}$$

Clearly assumptions (*) and (**) are not fulfilled by $\mathbf{K}(\theta_0)$ and N fails to be any Poisson process.

3 Examples

Throughout what follows, we put $\Upsilon := \mathbb{R}$, $T := \mathfrak{B}$, $\tilde{\Omega} := \Upsilon^{\mathbb{N}}$, $\Omega := \tilde{\Omega} \times \mathbb{R}^d$, $\tilde{\Sigma} := \mathfrak{B}_{\mathbb{N}}$ and $\Sigma := \mathfrak{B}(\Omega) = \mathfrak{B}_{\mathbb{N}} \otimes \mathfrak{B}_d$ for simplicity.

First, we describe a method for the construction of non-trivial probability spaces admitting extended MRPs with parameters Θ and h , generalizing in this way Example 5.5 from [8].

ex2

Example 3.1 Let μ be an arbitrary probability measure on \mathfrak{B}_d , $Q_n(\theta)$ be probability measures on \mathfrak{B} for all $n \in \mathbb{N}$ and for any fixed $\theta \in \mathbb{R}^d$, which is absolutely continuous with respect to Lebesgue measure λ on \mathfrak{B} . Moreover, suppose that there exists a measurable map $h : \mathbb{R}^d \rightarrow \mathbb{R}^k$ such that $Q_n(\theta) = \mathbf{K}(h(\theta))$ for any $n \in \mathbb{N}$, where for any $B \in \mathfrak{B}$ the function $\mathbf{K}(h(\bullet))(B) : \mathbb{R}^d \rightarrow \mathbb{R}$ is \mathfrak{B}_d -measurable and $\mathbf{K}(h(\theta))((0, \infty)) = 1$. It then follows that there exists a unique probability measure $\tilde{P}_\theta := \otimes_{n \in \mathbb{N}} Q_n(\theta)$ on $\tilde{\Sigma}$, and a sequence $\tilde{W} := \{\tilde{W}_n\}_{n \in \mathbb{N}}$ of \tilde{P}_θ -independent random variables on $(\tilde{\Omega}, \tilde{\Sigma})$ such that

$$\tilde{W}_n(\omega) = \omega_n = \tilde{\pi}_n(\omega) \quad \text{for each } \omega \in \tilde{\Omega} \quad \text{and } n \in \mathbb{N}$$

where $\tilde{\pi}_n : \tilde{\Omega} \rightarrow \mathbb{R}_+$ is the canonical projection, satisfying

$$(13) \quad (\tilde{P}_\theta)_{\tilde{W}_n} = Q_n(\theta) \quad \text{for all } n \in \mathbb{N}.$$

Define the set function $\tilde{P} : \tilde{\Sigma} \rightarrow \mathbb{R}$ by means of

$$\tilde{P}(E) := \int_{\mathbb{R}^d} \tilde{P}_\theta(E) \mu(d\theta), \quad \text{for any } E \in \tilde{\Sigma}.$$

Then \tilde{P} is a probability measure on $\tilde{\Sigma}$ and $\{\tilde{P}_\theta\}_{\theta \in \mathbb{R}^d}$ is a disintegration of \tilde{P} over μ . Denote by $\tilde{N} := \{\tilde{N}_t\}_{t \in \mathbb{R}_+}$ the counting process induced by \tilde{W} . Put $P(E) := \int \tilde{P}_\theta(E^\theta) \mu(d\theta)$, for each $E \in \tilde{\Sigma}$, where E^θ is the θ -section of E , and $P_\theta := \tilde{P}_\theta \otimes \delta_\theta$ for any $\theta \in \mathbb{R}^d$. Then P is a probability measure on Σ and according to [7], Proposition 3.5, $\{P_\theta\}_{\theta \in \mathbb{R}^d}$ is a disintegration of P over μ consistent with the canonical projection $\pi_{\mathbb{R}^d}$ from Ω onto \mathbb{R}^d (compare [8], Example 5.5).

Clearly, putting $\Theta := \pi_{\mathbb{R}^d}$ we get $P_\Theta = \mu$. Set $W_n := \tilde{W}_n \circ \pi_{\tilde{\Omega}}$ for any $n \in \mathbb{N}$. It then follows by (13)

$$(14) \quad (P_\theta)_{W_n} = Q_n(\theta) \quad \text{for all } n \in \mathbb{N} \quad \text{and } \theta \in \mathbb{R}^d,$$

implying that the sequence W is P_θ -independent for all $\theta \in \mathbb{R}^d$. The latter together with [6], Lemma 4.1, implies that the process W is P -conditionally independent. Furthermore, condition (14) together with Lemma 2.9 implies that for each $n \in \mathbb{N}$ the equality $P_{W_n|\Theta} = \mathbf{K}(h(\Theta))$ holds $P \upharpoonright \sigma(\Theta)$ -a.s. true. Thus the counting process N induced by W is an extended MRP with parameters Θ and h .

In the next examples it is shown that there exist non-trivial probability spaces satisfying all assumptions of Proposition 2.7 or Theorem 2.11, which allow us to check whether a $(P, \mathbf{K}(\Theta))$ -MRP or an extended MRP with parameters Θ and h is a Markov one.

For the next examples let $d = 1$.

exs

Examples 3.2 (a) Let $\mu = \mathbf{Ga}(\alpha, \beta)$, with $\alpha, \beta > 0$, be a probability measure on $\mathfrak{B}((0, \infty))$ and let $h : (0, \infty) \rightarrow (0, \infty)$ be defined by $h(\theta) := a\theta + b$ for any $\theta > 0$, where $a > 0$ and $b \geq 0$ are constants. Fix on arbitrary $\theta > 0$ and define the probability measures $Q_n(\theta)$ on \mathfrak{B} by means of $Q_n(\theta) := \mathbf{Exp}(h(\theta))$ for all $n \in \mathbb{N}$. It then follows by Example 3.1, that there exist a map $\Theta := \pi_{(0, \infty)}$ from $\Omega := \mathbb{R}^{\mathbb{N}} \times (0, \infty)$ onto $(0, \infty)$, $\{P_\theta\}_{\theta > 0}$, P and a counting process N , such that the induced claim interarrival process W satisfies condition $(P_\theta)_{W_n} = Q_n(\theta)$ for all $n \in \mathbb{N}$ if $\theta > 0$ is fixed.

Since for P_Θ -a.a. $\theta > 0$ the density $f_{h(\theta)}$ defined by $f_{h(\theta)}(t) = (a\theta + b) \cdot e^{-(a\theta + b)t}$ is dominated for any $t > 0$ by the map $C \in \mathcal{L}^1(P_\Theta)$, defined by $C(\theta) = a\theta + b$, we obtain that assumptions $(*)$ and $(**)$ of Theorem 2.11 are satisfied.

Put $\hat{\Theta} = \check{q} \circ \Theta$ and $Q_{\hat{\theta}}(A) := (P_\bullet(A) \circ \check{q}^{-1})(\hat{\theta})$ for any $\hat{\theta} > b$ and $A \in \Sigma$.

It then follows by Lemma 2.4 that $\{Q_{\hat{\theta}}\}_{\hat{\theta} > b}$ is a disintegration of P over $P_{\hat{\Theta}}$ consistent with $\hat{\Theta}$, condition $(Q_{\hat{\theta}})_{W_n} = \mathbf{Exp}(\hat{\theta})$ holds true for any $n \in \mathbb{N}$ and $\hat{\theta} > b$, and the process W is $Q_{\hat{\theta}}$ -independent. But the latter implies that N is $Q_{\hat{\theta}}$ -PP($\hat{\theta}$) for any $\hat{\theta} > b$ (cf. e.g. [10], Theorem 2.3.4). Thus, according to [6], Proposition 4.4, we deduce that N is a P -MPP($\hat{\Theta}$) implying together with Theorem 2.11 that N satisfies each of the equivalent conditions (i) and (ii) of Theorem 2.11.

(b) Let $\mu = \mathbf{Ga}(\alpha, \beta)$, with $\alpha, \beta > 0$, be a probability measure on \mathfrak{B} and let $h : (0, \infty) \rightarrow (0, \infty)$ be defined by $h(\theta) = \frac{1}{\theta}$ for any $\theta > 0$. For any fixed $\theta > 0$ take $Q_n(\theta) = \mathbf{Par}(h(\theta), 1)$ for all $n \in \mathbb{N}$, i.e.

$$Q_n(\theta)(B) := \int_B \theta \cdot \left(\frac{1/\theta}{1/\theta + t} \right)^2 \cdot \chi_{(0, \infty)}(t) \lambda(dt) \quad \forall B \in \mathfrak{B}.$$

It then follows by Example 3.1, that there exist a map $\Theta := \pi_{(0, \infty)}$ from $\Omega := \mathbb{R}^{\mathbb{N}} \times (0, \infty)$ onto $(0, \infty)$, $\{P_\theta\}_{\theta > 0}$, P and a counting process N , such that the induced claim interarrival process W satisfies condition $(P_\theta)_{W_n} = Q_n(\theta)$ for all $n \in \mathbb{N}$ if $\theta > 0$ is fixed.

Since for P_Θ -a.a. $\theta > 0$ the density $f_{h(\theta)}$ defined by $f_{h(\theta)}(t) = \theta \cdot \left(\frac{1/\theta}{1/\theta + t} \right)^2$ is dominated for any $t > 0$ by the map $C \in \mathcal{L}^1(P_\Theta)$, defined by $C(\theta) = \theta$, we obtain that assumptions $(*)$ and $(**)$ of Theorem 2.11 are satisfied.

Put $\hat{\Theta} = \check{q} \circ \Theta$ and $Q_{\hat{\theta}}(A) := (P_\bullet(A) \circ \check{q}^{-1})(\hat{\theta})$ for any $\hat{\theta} > 0$ and $A \in \Sigma$.

Assume, if possible, that N is a Markov process. It then follows by Theorem 2.11 that N is a P -MPP($\hat{\Theta}$) implying by [6], Proposition 4.5, that W is $Q_{\hat{\theta}}$ -exponentially distributed with parameter $\hat{\theta}$ for $P_{\hat{\Theta}}$ -a.a. $\hat{\theta} > 0$, a contradiction, since the process W is a $Q_{\hat{\theta}}$ - $\mathbf{Par}(\hat{\theta}, 1)$ for $P_{\hat{\Theta}}$ -a.a. $\hat{\theta} > 0$.

(c) Let $\mu = \mathbf{U}(1, 2)$ be a probability measure on \mathfrak{B} and let $h : (1, 2) \longrightarrow (1, 2)$ be defined by $h(\theta) = \theta$ for any $\theta > 0$. Fix on arbitrary $\theta > 0$ and define the probability measures $Q_n(\theta)$ by means of $Q_n(\theta) = \mathbf{Ga}(\theta, 2, 1/2)$, for all $n \in \mathbb{N}$, i.e.

$$Q_n(\theta)(B) := \int_B \frac{1}{2 \cdot \theta} \cdot e^{-(\frac{t}{\theta})^{1/2}} \cdot \chi_{(0, \infty)}(t) \lambda(dt) \quad \forall B \in \mathfrak{B}.$$

It then follows by Example 3.1, that there exists a map $\Theta := \pi_{(1, 2)}$ from $\Omega := \mathbb{R}^{\mathbb{N}} \times (1, 2)$ onto $(1, 2)$, $\{P_\theta\}_{\theta > 0}$, P and a counting process N , the claim interarrival process W of which satisfies condition $(P_\theta)_{W_n} = Q_n(\theta)$ for all $n \in \mathbb{N}$ if $\theta > 0$ is fixed.

Since for P_Θ -a.a. $\theta > 0$ the density f_θ defined by $f_\theta(t) = \frac{1}{2 \cdot \theta} \cdot e^{-(\frac{t}{\theta})^{1/2}}$ is bounded for any $t > 0$ by the positive number $\frac{1}{2}$, we obtain that assumptions (*) and (**) of Proposition 2.7 are satisfied.

Put $\tilde{\Theta} = \check{p} \circ \Theta$ and $Q_{\tilde{\theta}}(A) := (P_\bullet(A) \circ \check{p}^{-1})(\tilde{\theta})$ for any $\tilde{\theta} \in (\frac{1}{4}, \frac{1}{2})$ and $A \in \Sigma$.

Assume, if possible, that N is a Markov process. It then follows by Theorem 2.11 that N is a P -MPP($\tilde{\Theta}$) implying by [6], Proposition 4.5, that W is $Q_{\tilde{\theta}}$ -exponentially distributed with parameter $\tilde{\theta}$ for $P_{\tilde{\Theta}}$ -a.a. $\tilde{\theta} \in (\frac{1}{4}, \frac{1}{2})$, a contradiction, since the process W is a $Q_{\tilde{\theta}}$ -**Gamma**($\tilde{\theta}, \beta, \gamma$) for $P_{\tilde{\Theta}}$ -a.a. $\tilde{\theta} \in (\frac{1}{4}, \frac{1}{2})$.

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